The road to figuring out the mean

Why adding up and dividing is not obvious

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Trial of the Pyx

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- What is an acceptable irregularity in the weighing?

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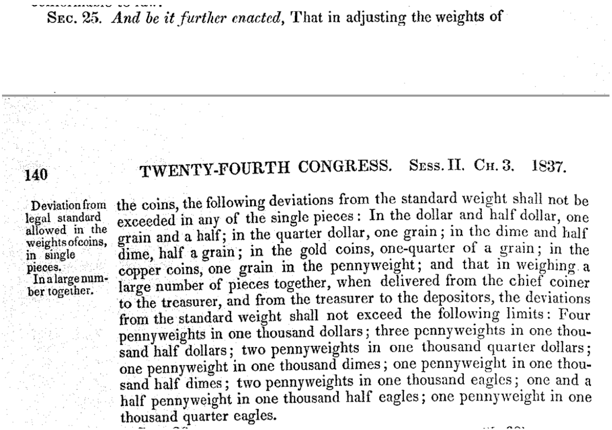
United States Congress (1837)



SEC. 25. *And be it further enacted*, That in adjusting the weights of

140 TWENTY-FOURTH CONGRESS. SESS. II. CH. 3. 1837.

Deviation from legal standard allowed in the weights of coins, in single pieces.

In a large number together.

the coins, the following deviations from the standard weight shall not be exceeded in any of the single pieces: In the dollar and half dollar, one grain and a half; in the quarter dollar, one grain; in the dime and half dime, half a grain; in the gold coins, one-quarter of a grain; in the copper coins, one grain in the pennyweight; and that in weighing a large number of pieces together, when delivered from the chief coiner to the treasurer, and from the treasurer to the depositors, the deviations from the standard weight shall not exceed the following limits: Four pennyweights in one thousand dollars; three pennyweights in one thousand half dollars; two pennyweights in one thousand quarter dollars; one pennyweight in one thousand dimes; one pennyweight in one thousand half dimes; two pennyweights in one thousand eagles; one and a half pennyweight in one thousand half eagles; one pennyweight in one thousand quarter eagles.

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Coin	Per coin	Per 1000 coins	Per coin $\times\sqrt{1000}$
Dollar	1.5	96	47.4
Half-dollar	1.5	72	47.4
Quarter	1	48	31.6
(Half-)Dime	0.5	24	15.8
Eagle ($10)	0.25	48	7.9

Table: Allowable error in grains

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People were concerned with quantifying and controlling error long before they had the tools and vocabulary.

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Thucydides (historian), c. 450 BCE: *"[The problem was for the Athenians] to force their way over the enemy's surrounding wall [...] Their method was as follows: they constructed ladders to reach the top of the enemy's wall, and they did this by calculating the height of the wall from the number of layers of bricks at a point which was facing in their direction and had not been plastered. The layers were counted by a lot of people at the same time, and though some were likely to get the figure wrong, the majority would get it right [...]. Thus, guessing what the thickness of a single brick was, they calculated how long their ladders would have to be"*

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Thucydides (historian), c. 450 BCE *"Homer gives the number of ships as 1,200 and says that the crew of each Boetian ship numbered 120, and the crews of Philoctetes were fifty men for each ship. By this, I imagine, he means to express the maximum and minimum of the various ships' companies [...] If, therefore, we reckon the number by taking an average of the biggest and smallest ships [...]"*

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ror herein. Exact trueth amongſt the vnconſtant waues of the ſea is not to bee looked for, though good inſtruments bee neuer ſo well applyed. Yet with heedfull diligence we may come ſo neare the trueth as the nature of the ſea, our ſight and inſtruments will ſuffer vs. Neither if there be diſagreement betwixt obſeruations, are they all by & by to be reiected? but as when many arrows are ſhot at a marke, and the marke afterwards taken away, hee may bee thought to worke according to reaſon, who to find out the place where the marke ſtood, ſhall ſeeke out the middle place amongſt all the arrowes : ſo amongſt many different obſeruations, the middlemoſt is likeſt to come neareſt the truth.

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On 1600 January 12/23 at 11h 50m the right ascension of Mars was:

	deg	min	sec
using the bright foot of Gemini	134	23	39
using Cor Leonis	134	27	37
using Pollux	134	23	18
at 12h 17m, using the third in the wing of Virgo	134	29	48
The mean, treating observations impartially	134	24	33

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Thomas Simpson, 1756: *"It is well known to your Lordship, that the method practiced by astronomers, in order to diminish the errors arising from the imperfections of instruments, and of the organs of sense, by taking the Mean of several observations, has not been generally received, but that some persons, of considerable note, have been of opinion, and even publickly maintained, that one single observation, taken with due care, was as much to be relied on as the Mean of a great number."*

Roger Cotes, 1722:

Roger Cotes, 1722: *"Let p be the place of some object defined by observation, q, r, s, the places of the same object from subsequent observations. Let there also be weights P, Q, R, S reciprocally proportional to the displacements which may arise from the errors in the single observations, and which are given from the given limits of error; and the weights P, Q, R, S are conceived as being placed at p, q, r, s, and their center of gravity Z is found: I say the point Z is the most probable place of the object, and may be safely had for its true place."*

Daniel Bernoulli, 1777:

Daniel Bernoulli, 1777: *"Astronomers as a class are men of the most scrupulous sagacity; it is to them therefore that I choose to propound these doubts that I have sometimes entertained about the universally accepted rule for handling several slightly discrepant observations of the same event. By these rules the observations are added together and the sum divided by the number of observations; the quotient is then accepted as the true value of the required quantity, until better and more certain information is obtained. In this way, if the several observations can be considered as having, as it were, the same weight, the center of gravity is accepted as the true position of the object under investigation."*

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The last assumption is that the MLE of a sample is its mean.

Gauss, 1809: *"It has been customary certainly to regard as an axiom the hypothesis that if any quantity has been determined by several direct observations, made under the same circumstances and with equal care, the arithmetical mean of the observed values affords the most probable value, if not rigorously, yet very nearly at least, so that it is always most safe to adhere to it."*

Conclusion: The normal distribution seals the mean

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- Central limit theorem is proved in greater and greater generality.
- "Standard law of the facility of errors"

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Gauss's proof (in modern notation)

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$$L(\theta) = \prod_{i=1}^{n} \varphi(x_i - \theta),$$

$$L'(\theta) = -\sum_{i=1}^{n} \varphi'(x_i - \theta) \prod_{j \neq i} \varphi(x_j - \theta)$$

$$= -L(\theta)\Big(\sum_{i=1}^{n} \frac{\varphi'(x_i - \theta)}{\varphi(x_i - \theta)} \Big)$$

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$$

By symmetry of φ, we have that $f(-x) = -f(x)$.

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Our previous equation is then

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where in the last step we use the fact that φ is maximised at 0.